Group cohomology

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1 Introduction

Group cohomology takes a group G and a G-module M and attaches to them a sequence of abelian groups $\mathrm{H}^i(G, M)$ for $i \geqslant 0$. Here's a simple example of what group cohomology can tell us.

For a G-module M, we can look at the submodule of G-**invariants**:

$$M^G = \{m \in M \mid mg = m \, \forall g \in G\}.$$

If A and B are G-modules, we can ask when it is true that

$$\left(\frac{A}{B}\right)^G \cong \frac{A^G}{B^G}.$$

In a while, we'll see that this holds if $\mathrm{H}^1(G, B) = 0$. Moreover, $\mathrm{H}^1(G, B)$, in some sense, gives us a measure of how badly this fails.

Group cohomology takes a while to define but, as with the real numbers, it is the properties that are important, not so much the definition. I'll introduce the properties that we need as I go along, but here is one particularly important one.

If we have an exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of G-modules, we can show that the sequence

$$0 \longrightarrow A^G \longrightarrow B^G \longrightarrow C^G$$

is always exact. However, we can do more: we can continue this sequence using cohomology groups as follows.

$$0 \to A^G \to B^G \to C^G \to \mathrm{H}^1(G, A) \to \mathrm{H}^1(G, B) \to \mathrm{H}^1(G, C) \to \mathrm{H}^2(G, A) \to \cdots$$

If we take this property on trust, we can already see how to resolve the problem in the introduction. We were looking at the short exact sequence

$$0 \longrightarrow B \longrightarrow A \longrightarrow \frac{A}{B} \longrightarrow 0.$$

Now look at the corresponding long exact sequence, namely

$$0 \longrightarrow B^G \longrightarrow A^G \longrightarrow \left(\frac{A}{B}\right)^G \longrightarrow \mathrm{H}^1(G, B).$$

We immediately see that if $\mathrm{H}^1(G, B) = 0$, then the sequence

$$0 \longrightarrow B^G \longrightarrow A^G \longrightarrow \left(\frac{A}{B}\right)^G \longrightarrow 0$$

is exact, which is what we wanted!

2 Induction and coinduction

We'd like to know what we can say about the cohomology of a subgroup H if we know about the cohomology of the big group G. The first thing we need is a way to get from H-modules to G-modules, so let M be an H-module. Then there are two canonical ways to construct a G-module from it. Take a set T of right coset representatives for H in G. Then the group algebra is a direct sum

$$\mathbb{Z}G = \bigoplus_{t \in T} \mathbb{Z}Ht.$$

Inspired by this, we can define the **induced module**,

$$M{\uparrow}_H^G = \bigoplus_{t \in T} Mt.$$

The action of G on $M{\uparrow}_H^G$ is given by writing an element $g \in G$ as ht for some $h \in H$ and $t \in T$ and then acting on $mt \in Mt$ by $(mt).g = mh't' \in Mt'$, where $th = h't'$, $h' \in H$, $t' \in T$. (More briefly, this is $M \otimes_{\mathbb{Z}H} \mathbb{Z}G$.)

There's a slightly different way to do this, and this is to use a product instead of a sum; this is called **coinduction**.

$$M{\Uparrow}_H^G = \prod_{t \in T} Mt,$$

with the action defined similarly. (Again, more briefly, this is $\mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M)$.)

The result about cohomology that we want is called **Shapiro's Lemma**. It says that if H is a subgroup of G and M is an H-module, then for each $i \geqslant 0$,

$$\mathrm{H}^i(H, M) \cong \mathrm{H}^i(G, M{\Uparrow}_H^G).$$

Of course, if H has finite index in G, then induction and coinduction coincide.

Let's look at a quick example. Let G be finite and consider $\mathrm{H}^i(G, \mathbb{Z}G)$. Notice that

$$\mathbb{Z}G = \bigoplus_{g \in G} \mathbb{Z}g.$$

However, since G is finite,

$$\bigoplus_{g \in G} \mathbb{Z}g = \prod_{g \in G} \mathbb{Z}g = \mathbb{Z}{\Uparrow}_1^G.$$

Therefore, by Shapiro's Lemma,

$$\mathrm{H}^i(G, \mathbb{Z}G) = \mathrm{H}^i(G, \mathbb{Z}{\Uparrow}_1^G) \cong \mathrm{H}^i(1, \mathbb{Z}) = 0,$$

for $i > 0$. We've just used another property of cohomology, namely that the cohomology of the trivial group is trivial in dimensions greater than 1. (In fact, a slight generalisation of this argument shows that for a finite group G, $\mathbb{Z}G$ is a self-injective algebra.)

We can also compute $\mathrm{H}^1(G, \mathbb{Z}G)$ for an infinite group G, but this is more complicated and turns out to be determined by the **number of ends** of G.

3 Cohomological dimension

Group cohomology was originally inspired by singular cohomology, so let's take a fact from singular cohomology and see what happens to it in group cohomology.

The fact that we're interested in is that if the space X has dimension d, then $\mathrm{H}^i(X, R) = 0$ for $i > d$.

Is something similar true for groups? The short answer is "no". For example, let's look at a finite cyclic group C_n. If we calculate its cohomology groups, we'll find that

$$\mathrm{H}^i(C_n, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ 0, & \text{if } n \text{ is odd}, \\ \mathbb{Z}/n\mathbb{Z}, & \text{if } n \text{ is even}. \end{cases}$$

Clearly there is no dimension d after which all cohomology groups are zero. Are there, then, groups for which there is such a d?

Let's define precisely what we mean: we say that a group G has **cohomological dimension** d, written $\operatorname{cd} G = d$, if there is a G-module M such that $\mathrm{H}^d(G, M) \neq 0$, but $\mathrm{H}^i(G, N) = 0$ for $i > d$ and for all G-modules N. If no such d exists, we write $\operatorname{cd} G = \infty$.

Let's assume that groups of finite cohomological dimension exist, and take G to be such a group. Suppose that $\operatorname{cd} G = d$. Let H be a subgroup of G and M be an H-module. Then by Shapiro's Lemma, above, we have

$$\mathrm{H}^i(H, M) \cong \mathrm{H}^i(G, M{\Uparrow}_H^G) = 0,$$

for $i > d$. Hence, H must also be finite-dimensional, of dimension at most d.

Now if g is an element of our finite-dimensional group G, consider $H = \langle g \rangle$. This is cyclic, either of finite or infinite order. But $\operatorname{cd} H \leqslant \operatorname{cd} G$, so H can't be finite, by the calculation above. That leaves only the possibility $H \cong \mathbb{Z}$, so that g has infinite order. Therefore, if G has finite cohomological dimension, it must be torsion-free. This certainly rules out finite groups!

Now let's look at $\mathbb{Z}$. We can calculate the cohomology explicitly, and we find that

$$\mathrm{H}^0(\mathbb{Z}, M) \cong M^{\mathbb{Z}}, \qquad \mathrm{H}^1(\mathbb{Z}, M) \cong M_{\mathbb{Z}}, \qquad \mathrm{H}^i(\mathbb{Z}, M) = 0 \text{ for } i > 1.$$

where M_G is module of **coinvariants** of M, the largest G-trivial quotient of M,

$$M_G = \frac{M}{(m - mg \mid m \in M,\, g \in G)}.$$

Therefore, $\operatorname{cd} \mathbb{Z} = 1$.

It's worthwhile to note that the fact that $\mathbb{Z}$ has dimension 1 is no coincidence. $\mathbb{Z}$ is a free group on one generator, and we can show that a free group on any number of generators must have dimension 1. Moreover, it has been shown by Stallings and Swan that any group of dimension 1 is free.

We can therefore think of cohomological dimension as being a measure of "how much space you have to move around in the group". Finite groups are very small and have "little space to move around", so these have infinite dimension, whereas free groups are very large and have no relations which, in a sense, means that you can do anything inside them, so these have dimension 1.

An example of a fairly large class of groups of finite dimension is the class of torsion-free polycyclic groups. For these groups, we can show that the cohomological dimension coincides with the Hirsch length.

4 Duality

If you're familiar with singular cohomology, you'd expect there to be a homology theory for groups, and, in fact, there is. Cohomology measures to what extent taking invariants fails to be exact, and homology does the same for coinvariants. That is, if we start with a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of G-modules, there is a long exact sequence

$$\cdots \to \mathrm{H}_2(G, C) \to \mathrm{H}_1(G, A) \to \mathrm{H}_1(G, B) \to \mathrm{H}_1(G, C) \to A_G \to B_G \to C_G \to 0.$$

Also, Shapiro's Lemma works for homology, with coinduction replaced by induction.

Now, going back to the example of $\mathbb{Z}$, if we calculate its homology, we find that

$$\mathrm{H}_0(\mathbb{Z}, M) \cong M_{\mathbb{Z}}, \qquad \mathrm{H}_1(\mathbb{Z}, M) \cong M^{\mathbb{Z}}, \qquad \mathrm{H}_i(\mathbb{Z}, M) = 0 \text{ for } i > 1.$$

Compare this to the cohomology of $\mathbb{Z}$: the groups are the same, but the order is reversed. That is, we have

$$\mathrm{H}^i(\mathbb{Z}, M) \cong \mathrm{H}_{1-i}(\mathbb{Z}, M)$$

for $0 \leqslant i \leqslant 1 = \operatorname{cd} \mathbb{Z}$. Again, this is no coincidence, but an example of **Poincaré duality**.

In general, we say that G is a **Poincaré duality group** of dimension d if it satisfies the following properties.

- $\operatorname{cd} G = d$ is finite,
- $\mathrm{H}^d(G, \mathbb{Z}G) \cong \mathbb{Z}$,
- $\mathrm{H}^i(G, \mathbb{Z}G) = 0$ for $0 \leqslant i \leqslant n-1$.

If G is a Poincaré duality group of dimension d, it satisfies

$$\mathrm{H}^i(G, M) \cong \mathrm{H}_{d-i}(G, M).$$

There are two ways to prove this. One is to use the abstract nonsense results from Grothendieck's Tohoku paper, the other is to use the cap product.

The class of groups we mentioned above, torsion-free polycyclic groups, also satisfies the requirements of Poincaré duality.

5 An example: the Heisenberg group

To finish off, let's look at a concrete example, the Heisenberg group. This is

$$G = \langle x,\, y,\, z \mid [x,y] = z,\, [x,z] = [y,z] = 1 \rangle \cong \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix},$$

and it's also the free nilpotent group of class two on two generators.

Suppose that we want to calculate the group of outer derivations of its group algebra, that is, linear maps $\mathbb{Z}G \to \mathbb{Z}G$ satisfying the Leibniz identity, modulo those which are given by commutation with an element of $\mathbb{Z}G$. This is precisely the first Hochschild cohomology group, $\mathrm{HH}^1(\mathbb{Z}G)$, of $\mathbb{Z}G$. We certainly won't worry about what Hochschild cohomology is here, we'll just need to know that

$$\mathrm{HH}^1(\mathbb{Z}G) \cong \bigoplus_{g \in_G G} \mathrm{H}^1(G, \mathbb{Z}{\uparrow}_{\mathrm{C}_G(g)}^G),$$

where $\mathrm{C}_G(g)$ is the centraliser of g in G and "$g \in_G G$" denotes a sum over conjugacy classes of G.

Our G is torsion-free polycyclic, its centre is a normal cyclic subgroup generated by z, with the quotient isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, generated by the images of x and y. It therefore has Hirsch length, and hence cohomological dimension, 3.

It has two kinds of centralisers: centralisers of central elements which are, of course, the whole group, and centralisers of non-central elements which all have Hirsch length 2. For the first case, we're interested in the group $\mathrm{H}^1(G, \mathbb{Z})$ (induction from G to itself does nothing), and we can, in fact, calculate this for any group G.

Consider the augmentation short exact sequence

$$0 \longrightarrow I_G \longrightarrow \mathbb{Z}G \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where I_G is the augmentation ideal $\mathrm{Ker}(\sum_{g\in G} a_g g \mapsto \sum_{g \in G} a_g)$. We use a generalisation of group cohomology and consider what happens when we apply the (contravariant) functor $\mathrm{Hom}_{\mathbb{Z}G}(-, \mathbb{Z})$ to the short exact sequence. We shall get a long exact sequence with H^i replaced by $\mathrm{Ext}^i_{\mathbb{Z}G}$.

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z} \longrightarrow \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, \mathbb{Z}) \cong \mathbb{Z} \longrightarrow \mathrm{Hom}_{\mathbb{Z}G}(I_G, \mathbb{Z}) \\ \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}) = \mathrm{H}^1(G, \mathbb{Z}) \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}G}(\mathbb{Z}G, \mathbb{Z}) = 0. \end{aligned}$$

Now, the map $\mathbb{Z} \to \mathbb{Z}$ turns out to be the identity, so we can insert a zero after it in the sequence, and so we're left with

$$\mathrm{H}^1(G, \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}G}(I_G, \mathbb{Z}).$$

However, since $\mathbb{Z}$ is a trivial module, each map $I_G \to \mathbb{Z}$ factors through I_G/I_G^2, which is also a trivial module. Now, $I_G/I_G^2 \cong G/G'$, and so

$$\mathrm{H}^1(G, \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(G^{\mathrm{ab}}, \mathbb{Z}).$$

In the case of the Heisenberg group, this gives

$$\mathrm{H}^1(G, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

We're now left with calculating $\mathrm{H}^1(G, \mathbb{Z}{\uparrow}_H^G)$ when H has Hirsch length 2. To do this we use Poincaré duality and Shapiro's Lemma.

$$\mathrm{H}^1(G, \mathbb{Z}{\uparrow}_H^G) \cong \mathrm{H}_2(G, \mathbb{Z}{\uparrow}_H^G) \cong \mathrm{H}_2(H, \mathbb{Z}) \cong \mathrm{H}^0(H, \mathbb{Z}) \cong \mathbb{Z}_H = \mathbb{Z},$$

so we're done.

We've shown that there are two outer derivations corresponding to each central element and one outer derivation corresponding to each non-central conjugacy class.